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Letter to the Editor

Best Approximation on Convex Sets in a Metric Space

Communicated by E. W. Cheney

In this paper the concepts of strictly convex and uniformly convex normed linear spaces are extended to metric spaces. A relationship between strictly convex and uniformly convex metric spaces is established. Certain existence and uniqueness theorems on best approximation in these spaces as well as in a complete metric space are proved.

1. Introduction

The problem of existence and uniqueness of best approximation has been studied by many investigators. Recent publications, Cheney [1], Singer [2], Davis [3] and others contain some of the results on the subject. Reference 1 contains the result on the existence of best approximation on a compact set in a metric space whereas unicity has been proved by taking the space to be a normed linear space. The existence theorem on an approximatively compact set in a metric space has been discussed in [2]. The purpose of this paper is to prove existence and uniqueness theorems on best approximation in a strictly convex metric space, a uniformly convex metric space, and for an approximatively Cauchy set in a complete metric space.

2. Strictly Convex and Uniformly Convex Metric Spaces

Definition 1. A strongly convex [4] metric space (X, d) is said to be *strictly convex* if

$$d(x, x_0) \leqslant r,\ d(y, x_0) \leqslant r \text{ imply}$$
$$d(z, x_0) < r, \text{ unless } x = y,$$

where x_0 is arbitrary but fixed point of X, z is the midpoint of x and y, and r is any finite real number.

Definition 2. A strongly convex metric space (X, d) is said to be *uniformly convex* if there corresponds to each pair of positive numbers (ϵ, r)

a positive number δ such that $d(x, y) < \epsilon$ whenever $d(x, x_0) \leqslant d(y, x_0) \leqslant r < d(z, x_0) + \delta$, z being the midpoint of x and y, and the other points being arbitrary.

THEOREM 1. (a) *Every uniformly convex metric space is strictly convex.*

(b) *Every compact strictly convex metric space is uniformly convex.*

Proof. (a) is an immediate consequence of the above definitions.

(b) Let (X, d) be a strictly convex compact metric space and $\epsilon > 0$ be given.

Define : $S = \{\langle x, y\rangle : d(x, x_0) \leqslant r, d(y, x_0) \leqslant r$ and $d(x, y) \geqslant \epsilon$ where $x, y \in X$ and x_0 is arbitrary fixed point of $X\}$.

It can be shown that S is a closed subset of $X \times X$, the metric on $X \times X$ being

$$d_1(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle) = [\{d(x_1, x_2)\}^2 + \{d(y_1, y_2)\}^2]^{1/2}.$$

S, being a closed subset of a compact metric space, is compact.

Define $\phi : S \to R$ as

$\phi(\langle x, y\rangle) = r - d(z, x_0)$, where z is the midpoint of x and y.

ϕ, by the strict convexity of X, is a positive continuous real-valued function on a compact set S. It attains its positive infimum, say, δ on S.

Therefore, for $\langle x, y\rangle \in S$, we have

$$r - d(z, x_0) \geqslant \delta$$

or, in other words

$$d(z, x_0) > r - \delta \text{ implies } d(x, y) < \epsilon,$$

which implies that the space is uniformly convex.

3. EXISTENCE AND UNIQUENESS THEOREMS ON BEST APPROXIMATION

DEFINITION 3. A subset K of a metric space (X, d) is said to be *convex* if for any two points x, y, in K any point between them is also in K.

THEOREM 2. An approximatively compact [2] convex set K in a strictly convex metric sapce (X, d) is Tschebychev [6].

Proof. The existence has been discussed in [2]. For uniqueness, if x_1^*, x_2^*

are two points in K of minimum distance, say r, from an arbitrary point, p, of the space (X, d), then strict convexity of X implies

$$d(x^{**}, p) < r \text{ unless } x_1^* = x_2^*,$$

where $x^{**} \in K$ is the midpoint of x_1^* and x_2^*, a contradiction.

COROLLARY 1. *A boundedly compact* [2] *closed convex subset of a strictly convex metrix space is Tschebychev.*

This is a consequence of the fact that in a metric space a boundedly compact closed set is approximatively compact [2].

THEOREM 3. *A compact convex set K in a strictly convex metric space (X, d) is Tschebychev.*

This is a consequence of Theorem 2 and the fact that a compact set in a metric space is approximatively compact.

COROLLARY 2. *A closed convex subset of a strictly convex compact metric space is Tschebychev.*

THEOREM 4. *A complete convex set K in a uniformly convex metric space (X, d) is Tschebychev.*

The proof of this theorem is similar to that of Theorem 2.

COROLLARY 3. *A closed convex subset of a complete uniformly convex metric space is Tschebychev.*

DEFINITION 4. A set K in a metric space (X, d) is said to be *approximatively Cauchy's set* if every sequence $\langle x_n \rangle$ in K, satisfying.

$\lim_{n\to\infty} d(x_n, x_0) = \inf\{d(x, x_0): x \in K, x_0$ is arbitrary fixed point of $X\}$, is a Cauchy's sequence.

THEOREM 5. *A closed approximatively Cauchy's set K in a complete metric space (X, d) is Tschebychev.*

Proof. A closed approximatively Cauchy's set being approximatively compact, the existence follows from Theorem 2.

The uniqueness can be established by considering the sequence $\langle x_n \rangle$ defined as

$$x_n = \begin{cases} x_1^*, & \text{if } n \text{ is odd} \\ x_2^*, & \text{if } n \text{ is even} \end{cases}$$

where x_1^*, x_2^* are two points in K of minimum distance from an arbitrary point p of the space (X, d).

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